Calculus Placement Test

To receive the full benefit of this test, watch the student to ensure he has mastered the concepts presented in Calculus.
If he struggles with the material on this exam, he should begin in Calculus.

I. Let $f(x) = x^2$, $g(x) = 3^x$ and $H(x) = 1 - x^2$.

1. Find $f(g(2))$.

2. Find $f(x) + 2H(x)$.

3. Find $\lim_{x\to 0} g(x)$.

4. Find $\lim_{x\to\infty} \frac{H(x)}{f(x)}$.

5. Using the definition, find the first derivative of $f(x) = 2x^2$.

II. Find the first derivative for each of the following.

1. $r = \sqrt{1-2\theta}$; find $dr/d\theta$.

2. $w = xe^{2x}$; find dw/dx.

3. $y = \ln(\sin(\theta))$; find $dy/d\theta$.

III. Integrate the following:

1. $\int_{1}^{e^2} \frac{\sqrt{\ln(x)}}{x}\,dx =$

2. $\int \sin(2x)\cos(2x)dx =$

IV. Find all extrema and any inflection points for $f(x) = -2x^3 + 6x^2 - 3$. Find the regions of concavity. Include a sketch.

V. Find the area of the region bounded by the curve $y=\sqrt{x-1}$ and the lines y = 1 and y = 2.

VI. Evaluate the following limits.

1. $\lim_{x\to 3} \frac{x^2-9}{x^2+x} =$

2. $\lim_{x\to 3} \frac{x^2-9}{x-3} =$

3. $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)} =$

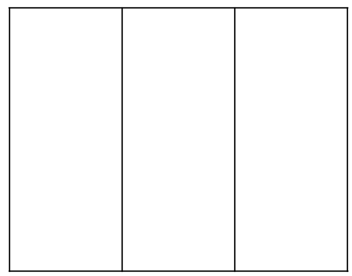
4. $\lim_{x\to \infty} \left(1+\frac{5}{x}\right)^{2x} =$

5. $\lim_{x \to \infty} \frac{3x^2 - 4x + 7}{x^2 + 3} =$

VII. Find the equation of the line tangent to $x^2 - \ln(x)$ at (1, 1).

VIII. Find the area between $y = x^2 + 1$ and $y = x + 3$. Include a sketch.

IX. Sixteen meters of fencing are to be used to make animal pens. Each of the three pens will be equal in size. Find the dimensions of the total area, which would be a maximum.



Solutions

I.

1. $f(g(x)) = f(3^x) = (3^x)^2 = 3^{2x}$

   $f(g(2)) = 3^{2(2)} = 3^4 = 81$

2. $f(x) + 2H(x) = x^2 + 2(1 - x^2)$

   $= x^2 + 2 - 2x^2$

   $= 2 - x^2$

3. $\lim_{x \to 0} 3^x = 3^0 = 1$

4. $\lim_{x \to \infty} \frac{H(x)}{f(x)} \lim_{x \to \infty} \frac{1 - x^2}{x^2} = \lim_{x \to \infty} \frac{\frac{1}{x^2} - 1}{1} = -1$

5. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

   $= \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$

   $= \lim_{h \to 0} \frac{(2x^2 + 2xh + h^2) - 2x^2}{h}$

   $= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$

   $= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$

   $= \lim_{h \to 0} (4x + 2h) = 4x$

II.

1. $r = \sqrt{1 - 2\theta}$ $\quad \frac{dr}{d\theta} = \frac{1}{2}(1 - 2\theta)^{-\frac{1}{2}}(-2)$

   $\frac{dr}{d\theta} = \frac{-1}{\sqrt{1 - 2\theta}}$

2. $w = xe^{2x}$ $\quad \frac{dw}{dx} = x(2e^{2x}) + e^{2x}(1)$

   $= e^{2x}(2x + 1)$

3. $y = \ln(\sin(\theta))$ $\quad \frac{dy}{d\theta} = \frac{1}{\sin(\theta)} \cdot \cos(\theta)$

   $= \cot(\theta)$

III.

1. Let $u = \ln(x)$; $du = \frac{1}{x}dx$

   $\int_1^{e^2} \frac{\sqrt{\ln(x)}}{x} dx = \int_1^{e^2} u^{\frac{1}{2}} du$

   $= \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{e^2}$

   $= \left. \frac{2}{3}(\ln(x))^{\frac{3}{2}} \right|_1^{e^2}$

   $= \frac{2}{3}\left[(\ln(e^2))^{\frac{3}{2}} - (\ln(1))^{\frac{3}{2}}\right]$

   $= \frac{2}{3}(\sqrt{8})$

   $= \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$

2. Let $u = \sin(2x)$; $du = \cos(2x) \cdot dx \cdot 2$

   $\frac{1}{2}\int \sin(2x)\cos(2x)dx \cdot 2$

   $= \frac{1}{2}\int u\,du$

   $= \frac{1}{2}\frac{u^2}{2} + C$

   $= \frac{1}{4}(\sin^2(2x)) + C$

IV. $f(x) = -2x^3 + 6x^2 - 3$

$f'(x) = -6x^2 + 12x$

$-6x(x - 2) = 0$

$x = 0$, $x = 2$ are critical points.

$f''(x) = -12x + 12$

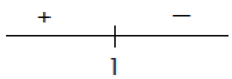
$-12(x - 1) = 0$

$x = 1$ possible inflection point

$f''(0) = 12$ so $x = 0$ is a min.

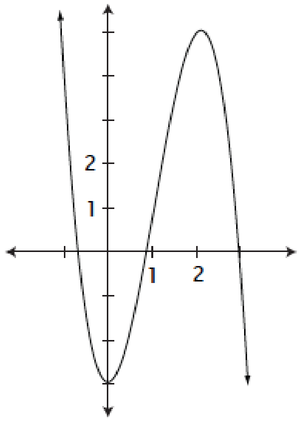
$f''(2) = -12$ so $x = 2$ is a max.

f''



$x = 1$ is an inflection point

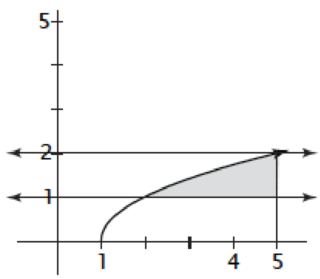
x	y
0	–3
1	1
2	5
3	–3
–1	5



f is concave up from $(-\infty, 1)$
and concave down from $(1, \infty)$.

V. $y = \sqrt{x-1}$

$y^2 = x - 1;\ x = y^2 + 1$



$$\int_1^2 (y^2+1)\,dy = \left(\frac{y^3}{3} + y\right)\Big|_1^2$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = \frac{10}{3}$$

VI.

1. $\lim_{x\to 3} \frac{x^2-9}{x^2+x} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x(x+1)} = 0$

2. $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = 6$

3. $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)} = \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)} = 0$ using LR

4. $\lim_{x\to\infty} \left(1 + \frac{5}{x}\right)^{2x}$

   $\frac{5}{x} \to 0$ so $\lim_{x\to\infty} 1^{2x} = 1$

5. $\lim_{x\to\infty} \frac{3x^2-4x+7}{x^2+3} = \lim_{x\to\infty} \frac{3 - \frac{4}{x} + \frac{7}{x^2}}{1 + \frac{3}{x^2}} = 3$

VII. $f(x) = x^2 - \ln(x)$

$f'(x) = 2x - \frac{1}{x}$

$f'(1) = 1$

$y = mx + b$

$1 = 1(1) + b$

$b = 0$

So the equation of the tangent line is $y = x$.

VIII. $y = x^2 + 1;\ y = x + 3$

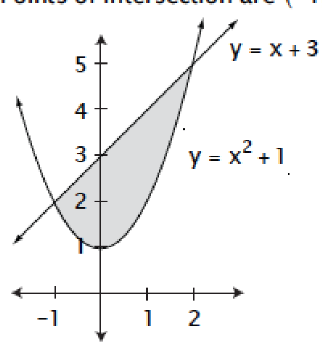
$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1,\ 2$

Points of intersection are $(-1, 2)$ and $(2, 5)$.



$$\int_{-1}^{2} [(x+3)-(x+1)]dx$$

$$= \int_{-1}^{2} (-x^2 + x + 2)dx$$

$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Bigg|_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= -\frac{8}{3} - \frac{5}{6} + 8 = 4\frac{1}{2}$$

IX. $2x + 4y = 16$

$$y = 4 - \frac{1}{2}x$$

$$A(x) = xy$$

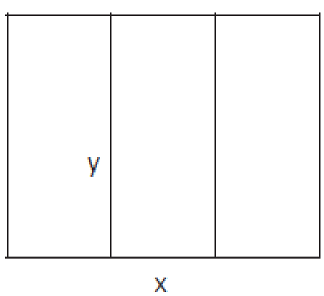
$$= x\left(4 - \frac{1}{2}x\right)$$

$$= 4x - \frac{1}{2}x^2$$

$A'(x) = 4 - x = 0$ when $x = 4$

$A''(x) = -1$

so $A''(4) = -1$ which yields a max.



If x = 4, when $y = 4 - \frac{1}{2}(4) = 2$.

The dimensions of the largest possible area would be 4 ft by 2 ft.